POISSON SUMMATION AND PERIODIZATION

PO-LAM YUNG

We give some heuristics for the Poisson summation formula via periodization, and provide an alternative proof that is slightly more motivated.

1. Some heuristics for the Poisson summation formula

The Poisson summation formula states that

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) \tag{1}$$

under suitable hypothesis on f, where

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx.$$

One way to understand it is that it gives a way of computing $\sum_{n\in\mathbb{Z}} f(n)$. But how can we come up with the right hand side, if we didn't know the answer ahead of time? A hint lies in the technique of periodization: maybe one should generalize, and ask for a formula for

$$\sum_{n\in\mathbb{Z}} f(x+n) \tag{2}$$

for all x; then we may set $x = 0$ to recover $\sum_{n\in\mathbb{Z}} f(n)$. Let $F(x)$ be the expression in (2). The key is to observe that it is a periodic function on $\mathbb{R}$ with period 1, so maybe we can expand it in terms of its Fourier series:

$$F(x) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n x}, \qquad a_n = \int_0^1 F(x) e^{-2\pi i n x} dx.$$

But then

$$\begin{aligned} a_n &= \int_0^1 \sum_{m\in\mathbb{Z}} f(x+m) e^{-2\pi i n x} dx \\ &= \int_0^1 \sum_{m\in\mathbb{Z}} f(x+m) e^{-2\pi i n (x+m)} dx \\ &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x} dx = \widehat{f}(n), \end{aligned}$$

so maybe

$$F(x) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{2\pi i n x},$$

i.e.

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{2\pi i n x}. \tag{3}$$

Setting $x = 0$ yields (1).

We emphasize that all these are heuristics only; they can be made rigorous, but one needs to put appropriate hypothesis on f.

2. Detour: Another application of periodization

As a detour, we adopt a similar point of view to compute the value of the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{4}$$

Indeed, maybe one should periodize, and look at

$$\sum_{n \in \mathbb{Z}} \frac{1}{(x+n)^2}$$

instead. One can then recover the desired sum (4), via

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \lim_{x \to 0} \left(\sum_{n \in \mathbb{Z}} \frac{1}{(x+n)^2} - \frac{1}{x^2} \right). \tag{5}$$

But it is actually easier to introduce a complex variable z in place of a real variable x: let's try to compute instead

$$\sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^2}$$

when $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$. Fix one such z. Then if we believe in the Poisson summation formula (1), we would let

$$f(x) = \frac{1}{(z+x)^2}$$

and compute $\widehat{f}(n)$: indeed a quick computation using contour integrals and residue theorem shows that

$$\widehat{f}(n) = \begin{cases} (-2\pi i)^2 n e^{2\pi i n z} & \text{if } n > 0 \\ 0 & \text{if } n \leq 0 \end{cases}$$

so if we believe in the Poisson summation formula (1), we would obtain

$$\sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^2} = \sum_{n=1}^{\infty} (-2\pi i)^2 n e^{2\pi i n z}.$$

The right hand side is the derivative of a convergent geometric series: indeed

$$\sum_{n=1}^{\infty} (-2\pi i)^2 n e^{2\pi i n z} = \sum_{n=1}^{\infty} -2\pi i \frac{d}{dz} e^{2\pi i n z} = -2\pi i \frac{d}{dz} \frac{e^{2\pi i z}}{1 - e^{2\pi i z}} = \frac{\pi^2}{\sin^2(\pi z)}.$$

Hence if the Poisson summation formula can be applied in this case, then we obtain a beautiful formula, namely

$$\sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^2} = \frac{\pi^2}{\sin^2(\pi z)}. \tag{6}$$

By analytic continuation, this would then hold for all $z \in \mathbb{C} \setminus \mathbb{Z}$. In view of (4), we may then compute the sum (4), by taking

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \lim_{x \to 0} \left(\frac{\pi^2}{\sin^2(\pi x)} - \frac{1}{x^2} \right) = \frac{\pi^2}{6}.$$

To fully justify (6), we may either make the Poisson summation formula rigorous for the function $f(x) = \frac{1}{(z+x)^2}$, or use a different argument. The most direct argument (once you guessed that (6) is true) is to use complex analysis again: let

$$H(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^2} - \frac{\pi^2}{\sin^2(\pi z)}.$$

Then H is holomorphic on $\mathbb{C} \setminus \mathbb{Z}$, bounded near the integers, and

$$\lim_{y \to \pm\infty} H(x+iy) = 0$$

for all $x \in \mathbb{R}$. (All these can be checked, with the help of the observation that H is periodic: $H(z+1) = H(z)$ for all $z \in \mathbb{C} \setminus \mathbb{Z}$.) Thus H is identically zero by Liouville's theorem. (See also the argument in Chapter 5.3.2 of [1].) This justifies (6) completely, and hence gives a full rigorous proof that $\sum_{n=1}^{\infty}(1/n^2)$ is $\pi^2/6$.

We note another consequence of (6): by taking its anti-derivative, we have

$$\sum_{n \in \mathbb{Z}} \frac{1}{z+n} = \pi \cot(\pi z) \tag{7}$$

for all $z \in \mathbb{C} \setminus \mathbb{Z}$. This identity must be interpreted carefully, for the sum on the left hand side does not converge absolutely. Nevertheless,

$$\sum_{n=-N}^{N} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{N} \left(\frac{1}{z+n} - \frac{1}{z-n} \right) = \frac{1}{z} + \sum_{n=1}^{N} \frac{2z}{z^2 - n^2}$$

converges absolutely for all $z \in \mathbb{C} \setminus \mathbb{Z}$ as $N \to \infty$, and this is the meaning we attach to the left hand side of (7).

Now let

$$G(z) = \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{z+n} - \pi \cot(\pi z).$$

Then G is holomorphic on $\mathbb{C} \setminus \mathbb{Z}$, and $G' = H = 0$ there, so G is a constant; by considering $\lim_{z \to 0} G(z)$, we see that G is identically zero. This proves (7) for all $z \in \mathbb{C} \setminus \mathbb{Z}$.

3. Proof of the Poisson summation formula

The version of Poisson summation formula we will prove is the following:

Theorem 1. *Suppose $f \colon \mathbb{R} \to \mathbb{C}$ admits a holomorphic extension to a horizontal strip $\{z \in \mathbb{C} \colon |Im\, z| < a\}$ for some $a > 0$, and that the holomorphic extension satisfies*

$$|f(z)| \leq \frac{A}{1+|z|^2}$$

for all z in the strip. Then

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n).$$

One way of doing it is to observe that $\widehat{f}(n)$ decays exponentially fast as $n \to \pm\infty$ (c.f. Theorem 2.1 in Chapter 4 of [1]). Thus both sides of (3) are continuous functions on $[0,1]$. They have the same Fourier coefficients, so by a result in Fourier analysis, they must be equal everywhere. Setting $x = 0$ yields the desired identity.

Here we will give another proof of Theorem 1, avoiding Fourier analysis (c.f. the proof of Theorem 2.4 in Chapter 4 of [1]). The idea is that $f(n)$ is just the residue of $\frac{f(z)}{z-n}$ at $z = n$: hence

$$\sum_{n\in\mathbb{Z}} f(n) = \lim_{N\to\infty} \sum_{n=-N}^{N} \text{Res}_{z=-n} \frac{f(z)}{z+n} = \lim_{N\to\infty} \frac{1}{2\pi i} \int_{\gamma_N} \sum_{n=-\infty}^{\infty} \frac{f(z)}{z+n} dz$$

where γ_N is the positively oriented rectangular contour, with vertices $\pm(N+\frac{1}{2}) \pm ib$ for some $0 < b < a$. In view of (7), we get

$$\sum_{n\in\mathbb{Z}} f(n) = \lim_{N\to\infty} \frac{1}{2i} \int_{\gamma_N} f(z)\cot(\pi z) dz.$$

(Technically we didn't really need to use (7); we can obtain the same identity, by just evaluating the right hand side using residue theorem.) Now the contributions from the vertical sides of the rectangle γ_N are negligible as $N \to \infty$. Thus we have

$$\sum_{n\in\mathbb{Z}} f(n) = \frac{1}{2i} \int_{L_1 - L_2} f(z)\cot(\pi z) dz, \tag{8}$$

where L_1 is the horizontal contour $\operatorname{Im} z = -b$, L_2 is the horizontal contour $\operatorname{Im} z = b$, both oriented in the positive x direction. But

$$\frac{1}{2i}\cot(\pi z) = \frac{e^{\pi i z} + e^{-\pi i z}}{2(e^{\pi i z} - e^{-\pi i z})}.$$

On L_1, we have $|e^{-\pi i z}| < 1$, so we should expand this as

$$\frac{1}{2i}\cot(\pi z) = \frac{1+e^{-2\pi i z}}{2} \frac{1}{1-e^{-2\pi i z}} = \frac{1+e^{-2\pi i z}}{2} \sum_{n=0}^{\infty} e^{-2\pi i n z} = \frac{1}{2} + \sum_{n=1}^{\infty} e^{-2\pi i n z}.$$

So

$$\frac{1}{2i} \int_{L_1} f(z)\cot(\pi z) dz = \int_{L_1} f(z) \left(\frac{1}{2} + \sum_{n=1}^{\infty} e^{-2\pi i n z} \right) dz. \tag{9}$$

Similarly, on L_2, we have $|e^{\pi i z}| < 1$, so we should use instead the expansion

$$\frac{1}{2i}\cot(\pi z) = \frac{e^{2\pi i z}+1}{2} \frac{1}{e^{2\pi i z}-1} = -\frac{e^{2\pi i z}+1}{2} \sum_{n=0}^{\infty} e^{2\pi i n z} = -\frac{1}{2} - \sum_{n=-\infty}^{-1} e^{-2\pi i n z},$$

and obtain

$$\frac{1}{2i} \int_{L_2} f(z)\cot(\pi z) dz = -\int_{L_2} f(z) \left(\frac{1}{2} + \sum_{n=-\infty}^{-1} e^{-2\pi i n z} \right) dz. \tag{10}$$

We now interchange the sum with the integral in both (9) and (10). This is justified since

$$\int_{L_1} |f(z)| \sum_{n=1}^{\infty} |e^{-2\pi i n z}||dz| = \int_{\mathbb{R}} |f(x-ib)| \sum_{n=1}^{\infty} e^{-2\pi n b} dx < \infty;$$

similarly

$$\int_{L_2} |f(z)| \sum_{n=-\infty}^{-1} |e^{-2\pi i n z}||dz| = \int_{\mathbb{R}} |f(x+ib)| \sum_{n=1}^{\infty} e^{-2\pi n b} dx < \infty.$$

Thus

$$\frac{1}{2i}\int_{L_1} f(z)\cot(\pi z)dz = \frac{1}{2}\int_{L_1} f(z)dz + \sum_{n=1}^{\infty}\int_{L_1} f(z)e^{-2\pi i n z}dz. \tag{11}$$

$$\frac{1}{2i}\int_{L_2} f(z)\cot(\pi z)dz = -\frac{1}{2}\int_{L_2} f(z)dz - \sum_{n=-\infty}^{-1}\int_{L_2} f(z)e^{-2\pi i n z}dz. \tag{12}$$

Shifting the contours L_1 and L_2 back to $\mathbb{R}$ in each of the terms on the right hand sides of (11) and (12), we get

$$\frac{1}{2i}\int_{L_1} f(z)\cot(\pi z)dz = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx + \sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f(x)e^{-2\pi i n x}dx = \frac{1}{2}\widehat{f}(0) + \sum_{n=1}^{\infty}\widehat{f}(n),$$

and

$$\frac{1}{2i}\int_{L_2} f(z)\cot(\pi z)dz = -\frac{1}{2}\int_{-\infty}^{\infty} f(x)dx - \sum_{n=-\infty}^{-1}\int_{-\infty}^{\infty} f(x)e^{-2\pi i n x}dx = -\frac{1}{2}\widehat{f}(0) - \sum_{n=-\infty}^{-1}\widehat{f}(n).$$

Thus in view of (8), we obtain the conclusion of our Theorem 1.

References

[1] Elias M. Stein and Rami Shakarchi, *Complex analysis*, Princeton Lectures in Analysis, vol. 2, Princeton University Press, Princeton, NJ, 2003.